Section B2: Methods of Solving for Equilibrium

There are many methods of projecting a time series based on finding a relationship with another time series. These methods, such as linear regression, require that the related series already be projected into the future. Economic theory suggests that energy prices are based primarily on long-term energy demand and production costs, while energy demand depends on energy prices, economic activity, technology efficiency, and other factors. One can therefore build a simple model of energy prices and consumption by developing a series of equations relating energy demand to prices and energy prices to demand, including other factors in the equations as needed. Methods based on such a model include Jacobi optimization, Gauss-Seidel optimization, and the Adaptive Expectation method. The National Energy Modeling System (NEMS) uses these methods as well as the method of Regula Falsi, described below.

B2.1. Jacobi Optimization

Jacobi Optimization is an iterative method of solving a set of linear equations by replacing the independent variables with previous solved-for values. The method begins with a set of educated guesses of the unknown values. We use these initial values to solve all of the equations in the set, and then we use the results to solve the equations a second time. We continue until the differences between the results of the successive iterations are small enough to fall within a predetermined tolerance or until a maximum number of iterations is completed.

As an example, consider the following equations, constructed as simple price and demand (linear) curves, with the quantity demanded declining as the price p increases and the price of the product supplied increasing as the quantity demanded q increases:

$$q = 5.6 - (0.3 \times p), \tag{B2.1.1}$$

and

$$p = 0.7 + (2.3 \times q). \tag{B2.1.2}$$

We start with an initial guess of $p = 0$ and $q = 0$, and solve both equations:

$$q = 5.6 - (0.3 \times 0) = 5.6, \tag{B2.1.3}$$

and

$$p = 0.7 + (2.3 \times 0) = 0.7. \tag{B2.1.4}$$

Next, we substitute $p = 0.7$ into the equation for q and $q = 5.6$ into the equation for p:

$$q = 5.6 - (0.3 \times 0.7) = 5.39, \tag{B2.1.5}$$

and

$$p = 0.7 + (2.3 \times 5.6) = 13.58. \tag{B2.1.6}$$

We continue substituting the resulting (p, q) pairs into the equations until we arrive in the neighborhood of the solution $(p = 8.0,\ q = 3.2)$.

B2.2. Gauss-Seidel Optimization

Much like Jacobi method, the Gauss-Seidel Optimization method is an iterative method of solving a set of linear equations by replacing the unknown variables with previously solved-for values and is initialized with educated guesses. The equations are arranged in a specified order, and each equation is solved in turn using the most recently available values. Since the equations are solved in sequence, the solution is at least partially dependent on the order in which the equations are solved. The accuracy of the initial guesses affects the speed of convergence.

The difference between the Gauss-Seidel method and the Jacobi method is that, in the Jacobi method, all equations are solved using the same set of data input, whereas, in the Gauss-Seidel method, each equation is solved using all data available at the time of its solution. The Gauss-Seidel method is most successful on a set of equations whose matrix representation is a strictly or irreducibly diagonally dominant or symmetric positive-definite matrix. If the matrix is not diagonally dominant, the method could result in a sequence of results that diverge rather than converge.

Using as an example the equations illustrating the Jacobi method, we have

$$q = 5.6 - (0.3 \times p), \tag{B2.2.1}$$

and

$$p = 0.7 + (2.3 \times q). \tag{B2.2.2}$$

We can start with an initial guess of $p = 0$ and $q = 0$. This time, we solve each equation in turn, using the latest available p and q values. We first set $p = 0$ and solve for q:

$$q = 5.6 - (0.3 \times 0) = 5.6 \tag{B2.2.3}$$

Next we set $q = 5.6$ and solve for p:

$$p = 0.7 + (2.3 \times 5.6) = 13.58 \tag{B2.2.4}$$

Then we set $p = 13.58$ and solve for q:

$$q = 5.6 - (0.3 \times 13.58) = 1.526 \tag{B2.2.5}$$

Similarly, we set $q = 1.526$ and solve for p:

$$p = 0.7 + (2.3 \times 1.526) = 4.2098 \tag{B2.2.6}$$

We continue substituting the resulting p and q values into the equations until we arrive in the neighborhood of the solution ($p = 8.0,\ q = 3.2$). Note that the Gauss-Seidel method converges more quickly than does the Jacobi method.

Convergence can be tested for each variable each time it is re-estimated. After each pass through the entire set of equations, overall convergence can be tested, and the process is stopped once all changes between iterations are within a set tolerance. We may also terminate the process after a fixed maximum number of iterations.

B2.3. Regula Falsi

Regula falsi, the method of false position, is a form of linear interpolation that can be used to iteratively solve a one-variable equation. Simple regula falsi may use direct proportion. If the equation be, for example,

$$182 = 4x + x/3, \tag{B2.3.1}$$

we may initially set $x = 6$ to obtain

$$4 \times 6 + 6/3 = 26. \tag{B2.3.2}$$

Because 26 is 7 times smaller than 182, we scale the initial guess ($x = 6$) up by a factor of 7 to arrive at the answer, $x = 42$. Clearly, equation B11.3.1 is also solvable by simple algebra.

Double false position can be used when the problem is more difficult and/or the equation is non-linear. Just as before, we guess a solution and calculate the result. We then revise the guess enough so that we are fairly confident that the second guess will err in the opposite direction from the first one, e.g., if the first guess, g_1 gave a solution a_1 that was too high, we try a second guess g_2 that gives a low solution, a_2. Once the problem has been bracketed between g_1 and g_2, our next guess could be of the form

$$g_3 = \frac{(a_1 \times g_2) - (a_2 \times g_1)}{(a_1 - a_2)}. \tag{B2.3.3}$$

The third trial should result in a point to one side of the answer, but closer. We then replace the old bracket point on that side with the newer point. If, continuing the example, the third guess g_3 be too low (like g_2), we would replace a_2 with a_3 in the equation above, and we would replace g_2 with g_3. If, however, g_2 be too high, we would replace a_1 with a_3 and g_1 with g_3. With enough iterations, we should converge on a solution, as long as the shape of the curve is either monotonically increasing or decreasing.

The key for solving problems using this method is to successfully bracket the answer quickly, i.e., have one guess that provides a too-high answer and a second guess that provides a too-low answer. Regula falsi works less well, or not at all, if the equations change across iterations.

B2.4. Adaptive Expectation Method

The basis for the adaptive expectation approach is the notion that increases in cumulative energy production would deplete domestic resources and thus place upward pressure on long-term energy prices. The following equation captures this general idea:

$$P_y = \left(A_y \times Q_y^e\right) + B_y$$

where P_y is the Henry Hub natural gas spot price for a future year, y, Q_y is the cumulative production from a specified starting year to year y, e is a judgment-specified parameter, and A_y and B_y are computed as explained below.

The approach was developed to represent the following assumptions:

- Prices should be upward sloping as a function of cumulative natural gas production, as prices could be expected to rise as existing resources are depleted.
- The rate of change in the natural gas spot price is a function of the economical reserves that remain to be discovered and produced. The value of the parameter *e* determines the shape of the function.

The approach assumes that, when cumulative natural gas production reaches a certain level QF, a target price PF results. In practice, the target production value PF is assumed, while QF and the annual production growth rates are model-based estimates. The parameters A_y and B_y are computed as follows:

$$A_y = \frac{PF - PS_{y-1}}{QF^e - QS_{y-1}^e},$$

and

$$B_y = PF - A_y \times QF^e,$$

where

D_{y-1}= natural gas production in year $y-1$;
PS_{y-1}= Henry Hub natural gas spot price in year $y-1$; and
QS_{y-1}= cumulative natural gas production from the starting year to year $y-1$.

The following equation extrapolates cumulative production for a future year y:

$$Q_y = Q_{y-1} + D_{y-1}$$

This generates the expected Henry Hub spot prices:

$$P_y = A_y \times Q_y^e + B_y$$

$$= PF + (Q_y^e - QF^e) \times (\frac{PF - PS_{y-1}}{QF^e - QS_{y-1}^e})$$

The value for *e* is assumed to be .70 until the price reaches a point at which the unconventional recovery of natural gas becomes economic ($3.50 in real 1998 dollars), and 1.3 afterward, creating an inflection point in the curve.